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RELATIVE ENDS AND DUALITY GROUPS

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We introduce a new algebraic invariant $\tilde{e}(G,S)$ for pairs of groups $S \le G$. It is related to the geometric end invariant of Houghton and Scott, but is more easily accessible to calculation by cohomological methods. We develop various techniques for computing $\tilde{e}(G,S)$ when G and S enjoy certain duality properties.

1. Introduction

Let G be any group and S any subgroup. Let $\mathscr{P}S$ denote the power set of G and let $\mathscr{F}_S G$ denote the set of *S-finite* subsets of G,

$$\mathscr{F}_S G := \{A \subseteq G \mid A \subseteq SF \text{ for some finite subset } F \text{ of } G\}.$$

Both $\mathscr{P}G$ and $\mathscr{F}_S G$ admit the action of G by right multiplication, and can be regarded as right G-modules over the field $\mathbb{F}$ of two elements. In analogy to the classical theory of ends of a group [6, 7, 15] we define an algebraic number of ends of the pair (G,S) in the following way.

Definition 1.1. $\tilde{e}(G,S) := \dim_{\mathbb{F}}(\mathscr{P}G/\mathscr{F}_S G)^G$.

This end invariant is implicit in our earlier work [9, 10, 11]. It is closely related to the geometric end invariant $e(G,S)$ introduced by Houghton [8] and Scott [12]. In the following section we shall also consider a common generalization to exhibit the similarities between the two definitions.

In this paper we concentrate on techniques for computing $\tilde{e}(G,S)$. Our method is based on the following simple lemma:

Lemma 1.2. *If S has infinite index in G, then* $\tilde{e}(G,S) = 1 + \dim_{\mathbb{F}} H^1(G, \mathscr{F}_S G)$.

This lemma is particularly useful when G and S enjoy certain duality properties; many examples can be found in Section 4 of this paper. In particular we shall show that $\tilde{e}(G,S)$ may be any natural number or infinite.

An important concept in our theory is the *commensurizer* of S in G, the set of $g\in G$ such that S and S^g are commensurable, which we denote by $\mathrm{Comm}_G(S)$. This is the largest subgroup of G which admits an action by left multiplication on $\mathcal{F}_S G$ and on $(\mathcal{P}G/\mathcal{F}_S G)^G$. Observe that $\mathrm{Comm}_G(S)$ only depends on the commensurability class of S, and that it contains every element of this class as a subgroup. In the third section we see that the theory of relative ends is very similar to the classical theory when $\mathrm{Comm}_G(S)$ is large enough. We shall prove the following theorem:

Theorem 1.3. *Let G and S be finitely generated and suppose that S has infinite index in* $\mathrm{Comm}_G(S)$. *Then $\tilde{e}(G,S)$ is either* 1, 2, *or infinite. In the case when $\tilde{e}(G,S)=2$ there are subgroups G_0 and S_0 of finite index in G and S, respectively, such that S_0 is normal in G_0 and G_0/S_0 is infinite cyclic.*

It would be very interesting to know whether there is an analogue of Stallings' Structure Theorem [14] for relative ends. Scott [12] has pointed out that when G splits over S, then $e(G,S)\geq 2$, and in [9] we also observe that the kernel of the restriction map

$$\mathrm{res}_S^G : H^1(G,\mathcal{F}_S G)\to H^1(S,\mathcal{F}_S G)$$

must be non-zero in this situation. We conjecture that for finitely generated groups G and S the non-vanishing of this kernel implies that G splits over some subgroup related to S.

2. Relative ends

Let M be a right $\mathbb{F}S$-module. There are two ways of constructing an $\mathbb{F}G$-module from M, by induction and coinduction:

$$\mathrm{Ind}_S^G M := \bigoplus_{g\in S\backslash G} Mg \cong M\otimes_{\mathbb{F}S}\mathbb{F}G,$$

$$\mathrm{Coind}_S^G M := \prod_{g\in S\backslash G} Mg \cong \mathrm{Hom}_{\mathbb{F}S}(\mathbb{F}G,M).$$

The inclusion of the direct sum into the Cartesian product induces an embedding of G-modules, natural in M,

$$j_S^G : \mathrm{Ind}_S^G M \to \mathrm{Coind}_S^G M,$$

whose cokernel we denote by $\mathrm{End}_S^G M$.

Since $\mathbb{F}G$ is free as an $\mathbb{F}S$-module, the functors Ind_S^G and Coind_S^G are exact, and so is End_S^G. If M is non-zero, then j_S^G is an isomorphism precisely if S has finite index in G. If S has infinite index in G, then $(\mathrm{Ind}_S^G M)^G=0$.

Definition 2.1. $e(G,S;M) := \dim_{\mathbb{F}}(\mathrm{End}_S^G M)^G$.

The elementary properties enjoyed by $e(G,S;M)$ are summarized in the following lemma:

Proposition 2.2. *Let $S \leq T$ be subgroups of G and let M be an S-module.*

(i) *If $|G:S| < \infty$, then $e(G,S;M) = 0$.*
(ii) *If $e(G,S;M) = 0$ and $M^S \neq 0$, then $|G:S| < \infty$.*
(iii) *If S is finitely generated and normal in G, then $e(G,S;M) = e(G/S, 1; M^S)$.*
(iv) *If $|G:T| < \infty$, then $e(G,S;M) = e(T,S;M)$.*
(v) *If $|G:T| = \infty$, then $e(G,S;M) = e(G,T;\mathrm{Coind}_S^T M)$.*
(vi) *If $|T:S| < \infty$, then $e(G,S;M) = e(G,T;\mathrm{Coind}_S^T M)$.*

Proof. Parts (i) and (ii) follow immediately from the above remarks. If S is normal in G, then we have an isomorphism of G/S-modules, natural in M,

$$H^i(S, \mathrm{Coind}_S^G M) \cong \mathrm{Coind}_1^{G/S} H^i(S,M) \quad \text{for } i = 0, 1, 2, \ldots.$$

If S is also finitely generated, then

$$H^i(S, \mathrm{Ind}_S^G M) \cong \mathrm{Ind}_1^{G/S} H^i(S,M) \quad \text{for } i = 0 \text{ and } 1.$$

In particular, the induced map $H^1(S, j_S^G)$ is injective, thus

$$(\mathrm{End}_S^G M)^S \cong \mathrm{End}_1^{G/S} M^S,$$

from which part (iii) follows.

To prove the remaining three parts of the proposition, we observe that the canonical isomorphism $\mathrm{Coind}_S^G M \cong \mathrm{Coind}_T^G \mathrm{Coind}_S^T M$ restricts to an embedding

$$i : \mathrm{Ind}_S^G M \to \mathrm{Ind}_T^G \mathrm{Coind}_S^T M.$$

Thus there is an epimorphism

$$p : \mathrm{End}_S^G M \to \mathrm{End}_T^G \mathrm{Coind}_S^T M,$$

with $\ker p \cong \mathrm{coker}\, i$. On the other hand, $\mathrm{Ind}_S^G M \cong \mathrm{Ind}_T^G \mathrm{Ind}_S^T M$, which means that $i = \mathrm{Ind}_T^G j_S^T$, so we have a short exact sequence

$$0 \to \mathrm{Ind}_T^G \mathrm{End}_S^T M \to \mathrm{End}_S^G M \to \mathrm{End}_T^G \mathrm{Coind}_S^T M \to 0.$$

Now if $|G:T| < \infty$, then $\mathrm{Ind}_T^G = \mathrm{Coind}_T^G$, therefore $\mathrm{End}_T^G = 0$ and

$$(\mathrm{End}_S^G M)^G \cong (\mathrm{Ind}_T^G \mathrm{End}_S^T M)^G \cong (\mathrm{Coind}_T^G \mathrm{End}_S^T M)^G \cong (\mathrm{End}_S^T M)^T,$$

by Shapiro's lemma, thus (iv) holds. If $|G:T| = \infty$, then $(\mathrm{Ind}_T^G \mathrm{End}_S^T M)^G = 0$, which implies (v). Finally, if $|T:S| < \infty$, then $\mathrm{End}_S^T M = 0$, and (vi) follows. □

In the simplest case, when S is the trivial subgroup and M equals $\mathbb{F}$, we obtain the classical number of ends of G. The geometric and algebraic number of ends of the pair (G, S) can also be seen as special cases of the general definition:

$$e(G,S)=e(G,S;\mathbb{F}), \qquad \tilde{e}(G,S)=e(G,S;\mathscr{P}S).$$

The next two lemmas describe the specializations of Proposition 2.2. to $e(G,S)$ and $\tilde{e}(G,S)$. For the proof of Lemma 2.3(iv) and (v) we refer to Scott's article [12].

Lemma 2.3. (i) $e(G,1)=e(G)$.

(ii) $e(G,S)=0$ *precisely if* $|G:S|<\infty$.

(iii) *If* $|G_1:G|<\infty$, *then* $e(G_1,S)=e(G,S)$.

(iv) *If N is a normal subgroup of G which is S-finite, then* $e(G,S)=e(G/N, SN/N)$.

(v) *If T contains S as a subgroup of finite index, then* $e(G,T)\le e(G,S)\le |T:S|e(G,T)$, *and these inequalities are the best possible.* □

Lemma 2.4. (i) $\tilde{e}(G,1)=e(G)$.

(ii) $\tilde{e}(G,S)=0$ *precisely if* $|G:S|<\infty$.

(iii) *If* $|G_1:G|<\infty$, *then* $\tilde{e}(G_1,S)=\tilde{e}(G,S)$.

(iv) *If S is finitely generated and normal in G, then* $\tilde{e}(G,S)=\tilde{e}(G/S)$.

(v) *If T contains S as a subgroup of finite index, then* $\tilde{e}(G,T)=\tilde{e}(G,S)$.

(vi) *If* $|G:T|=\infty$ *and* $S\le T$, *then* $\tilde{e}(G,S)\le\tilde{e}(G,T)$. □

Remarks. (1) Let G be the free group of rank two and S its commutator subgroup, then $\tilde{e}(G,S)=e(G)=\infty$, whereas $e(G,S)=e(G/S)=e(\mathbb{Z}\times\mathbb{Z})=1$. This shows that the condition that S be finitely generated cannot be omitted in Lemma 2.4(iv) above.

(2) Lemma 2.4(vi) shows that $\tilde{e}(G,S)$ only depends on the commensurability class of S in G. This also follows from the simple fact that S and T are commensurable subgroups of G if and only if $\mathscr{F}_S G=\mathscr{F}_T G$.

(3) A well-known theorem of Hopf [7] says that the classical number of ends of a group equals 0, 1, 2 or infinity. It should be noted that only for groups G with one end the number $\tilde{e}(G,S)$ may provide some interesting information about the embedding of S into G. If G has two ends, then any subgroup is either finite or has finite index in G, and if $e(G)=\infty$, then $\tilde{e}(G,S)=\infty$ for all subgroups S of infinite index.

(4) To prove Lemma 1.2 of the introduction assume that S has infinite index in G and consider the long exact sequence

$$0\to M^S\to(\mathrm{End}_S^G M)^G\to H^1(G,\mathrm{Ind}_S^G M)\to H^1(S,M)\to\cdots.$$

If M equals $\mathscr{P}S$, then $M^S\cong\mathbb{F}$ and $H^1(S,M)=0$ by Shapiro's Lemma.

The final lemma in this section describes the relation between the two different end invariants.

Lemma 2.5. (i) $e(G,S)\le\tilde{e}(G,S)$.

(ii) *If S is finitely generated and* $\tilde{e}(G,S)$ *is finite, then there exists a subgroup* S_0 *of finite index in S, such that* $e(G,S_0)=\tilde{e}(G,S_0)$.

Proof. The first statement follows from the fact that $\mathbb{F}$ embeds into $\mathscr{P}S$ as the S-invariant submodule, together with the left exactness of $(\mathrm{End}_S^G -)^G$.

For the proof of (ii) we consider the action of S by left multiplication. With respect to this action the module $\mathscr{F}_S G$ is a direct sum of copies of the coinduced module $\mathscr{P}S$, and if S is finitely generated, then $H^1(S, -)$, computed with the left action, commutes with direct sums. Now $H^1(S, \mathscr{F}_S G) = 0$, thus the sequence

$$0 \to {}^S(\mathscr{F}_S G) \to {}^S(\mathscr{P}G) \to {}^S(\mathrm{End}_S^G \mathscr{P}S) \to 0$$

is exact, which implies that ${}^S(\mathrm{End}_S^G \mathscr{P}S) \cong \mathrm{End}_S^G \mathbb{F}$. We assume that $\mathrm{End}_S^G \mathscr{P}S$ is finite as a set, so S has a subgroup S_0 of finite index which acts trivially on this module, and since the left action commutes with the right action it follows that $e(G, S_0) = \tilde{e}(G, S_0)$. □

3. The classical case

Throughout this section we assume that G is finitely generated, so we can consider the Cayley graph $\Gamma = \Gamma(G, X)$ of G with respect to some finite set X of generators. We shall need some well-known results concerning this. In the first, which was proved by Scott [12], we shall say that a subset C of G is *connected* if it is the vertex set of some connected subgraph of Γ.

Lemma 3.1. *If S is finitely generated, then every S-finite subset A of G is contained in some connected S-finite subset C of G.*

Proof. For the convenience of the reader, we record the basis of Scott's argument here. There exists a finite set D, such that $A \subseteq SD$, and a finite set E of generators of S. Now choose a finite connected set F containing $D \cup E \cup \{1\}$, then $C := SF$ satisfies the conditions of the lemma. □

For a subset C of G, set $\delta C := \bigcup_{x \in X} (C + Cx^{-1})$. If dC denotes the set of edges in Γ which have exactly one vertex in C, then δC is precisely the vertex set of dC. In the following version of a lemma of Cohen [3, Lemma 2.7] we say that two subsets A and B of G are *nested* if and only if at least one of the inclusions

$$A \subseteq B, \qquad A^c \subseteq B, \qquad A \subseteq B^c, \qquad A^c \subseteq B^c$$

holds.

Lemma 3.2. *Let A and B be subsets of G, and let C and D be connected subsets of G which contain δA and δB respectively. If $C \cap D = \emptyset$, then A and B are nested.* □

We say that a subset A of G is *S-almost invariant* if it represents an element of $(\mathscr{P}G/\mathscr{F}_S G)^G$, i.e. if $A + Ag$ is S-finite for all $g \in G$. Since G is finitely generated, the set A is S-almost invariant precisely if δA is S-finite. As a consequence of the previous two lemmas we obtain the following form of Scott's Lemma [12, Lemma 4.4].

Lemma 3.3. *If S and T are finitely generated subgroups of G, A is an S-almost invariant subset of G and B is a T-almost invariant subset of G, then there is a finite subset F of G such that for all g in $G \setminus SFT$, the subsets A and gB are nested.*

Proof. Since A is S-almost invariant, it follows that δA is S-finite, and we may choose, using Lemma 3.2, a connected S-finite subset of G containing δA. Similarly, there is a connected T-finite subset D of G containing δB. The left action of G on itself by multiplication gives rise to an action on Γ, and so if $g \in G$ is such that $C \cap gD = \emptyset$, then A and gB are nested, by Lemma 3.2. Thus we can take F to be any finite subset of G such that $CD^{-1} \subseteq SFT$. □

Now we come to the proof of Theorem 1.3. Here we shall combine Cohen's proof of the classical characterization of groups with two ends [3, Lemma 2.9] with the ideas used in the final stage of the proof of Theorem A in [9].

Proof of Theorem 1.3. We assume that $1 < \tilde{e}(G,S) < \infty$. Since $\tilde{e}(G,S) > 1$ we may choose some *proper* S-almost invariant set A in G, i.e. neither A nor A^c is S-finite. For each $g \in \mathrm{Comm}_G(S)$ the set gA is again S-almost invariant, and since $\tilde{e}(G,S)$ is finite, the subgroup

$$H := \{g \in \mathrm{Comm}_G(S) \mid gA + A \in \mathscr{F}_S G\}$$

has finite index in $\mathrm{Comm}_G(S)$. Nothing is lost if we replace S by $S \cap H$ and so assume that S is contained in H. This implies that $\tilde{e}(G,S) = e(G,S)$, therefore by varying A by an S-finite amount we can arrange that $A = SA$. Using Lemma 3.3 we may choose a finite subset F of H such that for all $g \in H \setminus SFS$ the subsets A and gA are nested. Since A is proper and $gA + A$ is S-finite this means that either $A \subseteq gA$ or $gA \subseteq A$. But the set of all $g \in G$ such that $gA = A$ is S-finite, therefore by enlarging the finite set F if necessary, we may assume that for all $g \in H \setminus SFS$ one of the strict inclusions

$$gA \subset A \quad \text{or} \quad A \subset gA$$

holds. Now F is a subset of $\mathrm{Comm}_G(S)$, therefore SFS is S-finite and there actually exists an element $g \in H \setminus SFS$. By interchanging A with its complement if necesary, we can arrange that for some element $a \in A$ we have

$$gA \subseteq A \setminus \{a\}.$$

Then $g^n A \subseteq A \setminus \{a\}$ for all positive integers n, thus the cyclic group Z generated by g is finite. Since we have arranged that $A = SA$, it also follows that $Z \cap S = 1$.

Next we claim that

$$\bigcap_{n\geq 1} g^n A = \emptyset.$$

Suppose that b is an element of that intersection, then for all $n \geq 1$ we have $b \in g^n A$. Therefore

$$g^{-n} \in Ab^{-1} \cap A^c a^{-1},$$

which is S-finite, but that implies that $Z \cap S$ is non-trivial, and we have reached a contradiction. It follows that

$$A = \bigcup_{n\geq 0} (g^n A \setminus g^{n+1} A) = \bigcup_{n\geq 0} g^n (A \setminus gA) \subseteq Z(A \setminus gA).$$

By a similar argument we can now show that $\bigcap_{n\geq 0} g^{-n} A^c = \emptyset$ and $A^c \subseteq Z(gA^c \setminus A^c)$. Hence

$$G = A \cup A^c \subseteq Z(gA + A),$$

and since g is an element of H, $gA + A$ is S-finite.

If z is any element of Z, then S^z is commensurable with S, and in particular each h in S^z has some power in S. It follows that S^z is contained in SFS. However, SFS is S-finite, so we conclude that $|S : S \cap S^z| \leq |S \setminus SFS|$ for each z. Since S is finitely generated, this bound on the indices of $S \cap S^z$ in S allows us to conclude that $S_0 := \bigcap_{z\in Z} S^z$ has finite index in S, and it is obviously normalized by Z. Now the group $G_0 := ZS_0$ has finite index in G, and so $\tilde{e}(G, S) = \tilde{e}(G_0, S_0) = e(Z) = 2$. □

Remark. Suppose that G and S are finitely generated and S has infinite index in its normalizer. Then Theorem 1.3 holds with $\tilde{e}(G, S)$ replaced by $e(G, S)$. This result has also been proved by Houghton [8].

Corollary 3.4. *Assume that G and S are finitely generated and that $\tilde{e}(G, S) = 2$. Then the set $\{T \leq G \mid T$ is commensurable with S and $e(G, T) = 2\}$ contains a unique maximal element $S^\dagger$. Furthermore, $N_G(S^\dagger) = \mathrm{Comm}_G(S^\dagger)$ and $N_G(S^\dagger)/S^\dagger$ is isomorphic to 1, C_2, C_∞ or $C_2 * C_2$.*

Proof. If S has finite index in its commensurizer C, then C is the unique maximal element of the commensurability class of S. Choose a proper S-almost invariant subset A of G, then the group

$$S^\dagger := \{g \in C \mid gA + A \in \mathcal{F}_S G\}$$

has index at most two in C. If T is any group commensurable with S, then $e(G, T) = 2$ holds precisely when T is contained in $S^\dagger$.

If S has infinite index in C, then we may choose subgroups G_0 and S_0 as in Theorem 1.3. Observe that C contains G_0 as a subgroup of finite index, therefore $S_1 := \bigcap_{g\in C} S_0^g$ has finite index in S_0 and is normal in C. The group C/S_1 has two ends, thus it contains a unique maximal normal finite subgroup $S^\dagger/S_1$; furthermore, $C/S^\dagger$ is isomorphic to C_∞ or $C_2 * C_2$. By Lemma 2.3 we have $e(G,S^\dagger)=e(C,S^\dagger)=e(C/S^\dagger)=2$.

In the case when $C/S^\dagger$ is infinite cyclic, the group $S^\dagger$ is the unique maximal element of the commensurability class of S. If $C/S^\dagger$ is isomorphic of $C_2 * C_2$, then any maximal subgroup T in this class contains $S^\dagger$ as a subgroup of index two, so $e(G,T)=e(C,T)=e(C_2 * C_2, C_2)=1$. In view of Lemma 2.3(iv) it is now clear that a subgroup T commensurable with S satisfies $e(G,T)=2$ if and only if T is contained in $S^\dagger$. □

4. Duality groups

We shall assume that the reader is familiar with the theory of duality groups [1,2], but to fix notations we make a few preliminary remarks. A group G is said to be of *type* FP over $\mathbb{F}$ if the trivial module $\mathbb{F}$ has a finite resolution

$$0 \to P_n \to \cdots \to P_0 \to \mathbb{F} \to 0$$

of finitely generated non-zero projective $\mathbb{F}G$-modules. It follows in particular that if $\operatorname{cd}_{\mathbb{F}} G = n$, then the left $\mathbb{F}G$-module $D_G := H^n(G, \mathbb{F}G)$ does not vanish. If, furthermore,

$$H^i(G, \mathbb{F}G) = 0 \quad \text{for } i = 0, \ldots, n-1,$$

then G is a *duality group* of dimension n, or short a D^n-group, and D_G is called the *dualizing module*, for there are natural isomorphisms for all integers i,

$$H^i(G, -) \cong H_{n-i}(G, - \otimes_{\mathbb{F}} D_G),$$

$$H_i(G, -) \cong H^{n-i}(G, \operatorname{Hom}_{\mathbb{F}}(D_G, -)),$$

where G acts diagonally on the tensor product and on the Hom-group. If the dualizing module D_G is isomorphic to $\mathbb{F}$, then G is called a Poincaré duality group, for short a PD^n-group. If G is just of type FP, then the duality isomorphisms exist at least for $i = n$.

Lemma 4.1. (i) *Suppose that G is a* D^n*-group and S is a subgroup of type* FP *with* $\operatorname{cd}_{\mathbb{F}} S = n-1$. *Then* $H^1(G, \mathscr{F}_S G) \cong \operatorname{Hom}_{\mathbb{F}S}(D_S, \mathscr{P}S \otimes D_G)$.

(ii) *If G is a* PD^n*-group, then* $\tilde{e}(G,S) = 1 + \dim_{\mathbb{F}} D_S$.

(iii) *If S is a* PD^{n-1}*-group, then* $\tilde{e}(G,S) - 1$ *equals the dimension of the largest locally finite-dimensional* $\mathbb{F}S$*-submodule of* D_G.

Proof. (i) Using the duality isomorphism for G we obtain

$$H^1(G, \mathscr{F}_S G) \cong H_{n-1}(G, \mathscr{F}_S G \otimes D_G).$$

It is well known that $(\mathrm{Ind}_S^G M)\otimes N \cong \mathrm{Ind}_S^G(M\otimes \mathrm{Res}_S^G N)$ for an S-module M and a G-module N, where the tensor products have diagonal G- and S-action, respectively. Now we have

$$H^1(G, \mathscr{F}_S G) \cong H_{n-1}(G, \mathrm{Ind}_S^G(\mathscr{P}S \otimes_{\mathbb{F}} D_G))$$

$$\cong H_{n-1}(S, \mathscr{P}S \otimes_{\mathbb{F}} D_G), \quad \text{by Shapiro's Lemma,}$$

$$\cong \mathrm{Hom}_{\mathbb{F}S}(D_S, \mathscr{P}S \otimes_{\mathbb{F}} D_G), \quad \text{by duality for } S.$$

(ii) If G is a PD^n-group, then $D_G \cong \mathbb{F}$, therefore

$$H^1(G, \mathscr{F}_S G) \cong \mathrm{Hom}_{\mathbb{F}S}(D_S, \mathscr{P}S) \cong \mathrm{Hom}_{\mathbb{F}}(D_S, \mathbb{F}).$$

(iii) Let S be a PD^{n-1}-group, then it follows from (i) that

$$H^1(G, \mathscr{F}_S G) \cong (\mathscr{P}S \otimes D_G)^S.$$

If $\mathscr{P}S \otimes D_G$ contains an element $\sum_{d\in D_G} A_d \otimes d$ which is fixed by S, then the $\mathbb{F}$-module spanned by the finite set $\{d \in D_G \mid A_d \neq \emptyset\}$ is invariant under the action of S. Assume now that M is a finite-dimensional $\mathbb{F}S$-module. Then the action of S factors through some finite quotient group Q, and

$$(\mathscr{P}S \otimes M)^S \cong (\mathscr{P}Q \otimes M)^Q \cong (\mathbb{F}Q \otimes M)^Q \cong \mathrm{Res}_1^S M.$$

The lemma now follows. □

A well-known theorem of Strebel [16] says that a subgroup S of a PD^n-group G has cohomological dimension n if and only if S has finite index in G. The following corollary to Lemma 4.1 may be seen as a supplement to Strebel's Theorem.

Corollary 4.2. *Let G be a PD^n-group and S a subgroup of type* FP. *Then* $\mathrm{cd}_{\mathbb{F}} S \le n-2$ *precisely if* $\tilde{e}(G,S)=1$.

Proof. The proof of Lemma 4.1 shows that $H^1(G, \mathscr{F}_S G)$ vanishes when $\mathrm{cd}_{\mathbb{F}} S \le n-2$ and is non-zero when $\mathrm{cd}_{\mathbb{F}} S = n-1$. If S has cohomological dimension n, then by Strebel's Theorem it also has finite index in G, and then $\tilde{e}(G,S)=0$. □

If S is a duality group, then by a theorem of Farrell [1, Theorem 9.8] the dualizing module D_S is either isomorphic to $\mathbb{F}$ or infinite dimensional as an $\mathbb{F}$-module. Thus we have

Corollary 4.3. *Let G be a PD^n-group and S a D^{n-1}-subgroup. Then either* $\tilde{e}(G,S)=\infty$ *or S is a PD^{n-1} group, in which case* $\tilde{e}(G,S)=2$. □

Proposition 4.4. *Let G be a torsion-free* PD^3*-group and S a subgroup of type* FP, *then one of the following holds*:

(0) $\tilde{e}(G,S)=0 \Leftrightarrow$ *S has a finite index in G*;

(1) $\tilde{e}(G,S)=1 \Leftrightarrow$ *S is a free group*;

(2) $\tilde{e}(G,S)=2 \Leftrightarrow$ *S is a* PD^2*-group*;

(∞) $\tilde{e}(G,S)=\infty$.

Proof. The equivalence (0) follows from Lemma 2.4(ii). By the Theorem of Dunwoody [5] a torsion-free group has cohomological dimension one over $\mathbb{F}$ if and only if it is a free group, so (1) follows from Corollary 4.2. It remains to consider the case when $1<\tilde{e}(G,S)<\infty$ and $\mathrm{cd}_{\mathbb{F}}\, S=2$. In view of Remark (2) in [1, §9.8] we may assume that S is a free product $S_1 * S_2$, where S_1 is a D^2-group. By the Mayer–Vietoris sequence for free products, $H_2(S,\mathscr{P}S)\cong H_2(S_1,\mathscr{P}S)\oplus H_2(S_2,\mathscr{P}S)$. Denoting the dualizing module of S_1 by D, we obtain

$$H_2(S_1,\mathscr{P}S)\cong \mathrm{Hom}_{\mathbb{F}S_1}(D,\mathscr{P}S)\cong \prod_{t\in S/S_1} \mathrm{Hom}_{\mathbb{F}}(D,t\,\mathbb{F}).$$

Recall that

$$\dim_{\mathbb{F}} H_2(S,\mathscr{P}S)=\dim_{\mathbb{F}} H^1(G,\mathscr{F}_S G)=\tilde{e}(G,S)-1,$$

which we assume to be finite. Therefore S_1 must have finite index in S, which means that S itself must be a duality group, and by Corollary 4.3 it follows that (2) holds. □

Remark. If G is a PD^2-group and S is any subgroup, then one of the following holds:

(0) $\tilde{e}(G,S)=0 \Leftrightarrow$ S has finite index in G;

(1) $\tilde{e}(G,S)=1 \Leftrightarrow$ S is the trivial group;

(2) $\tilde{e}(G,S)=2 \Leftrightarrow$ S is an infinite cyclic group;

(∞) $\tilde{e}(G,S)=\infty \Leftrightarrow$ S is a non-cyclic free group.

As another application of Lemma 4.1 we consider the one relator group $G:=\langle x,y \mid xy=yx^2\rangle$. This can be viewed as an ascending HNN-extension over the base group X, the infinite cyclic group generated by x, thus by [1, Proposition 9.16(b)], G is a D^2-group.

Proposition 4.5. *Let $G:=\langle x,y \mid xy=yx^2\rangle$ and $X:=\langle x\rangle$. For any subgroup S of G one of the following holds*:

(0) $\tilde{e}(G,S)=0 \Leftrightarrow$ *S has finite index in G*;

(1) $\tilde{e}(G,S)=1 \Leftrightarrow$ $S\cap X=1$;

(∞) $\tilde{e}(G,S)=\infty \Leftrightarrow$ *$S\cap X$ is non-trivial, and $|G:S|=\infty$.*

Proof. The dualizing module of G can be computed from the Mayer–Vietoris sequence or using Lyndon's resolution for one relator groups (see [1, Exercise to

§9.7]); it has the form $D_G = \mathbb{F}G/I$, where I is the left ideal generator by the Fox derivatives of the relator, i.e. $I := \mathbb{F}G(y+x+1) + \mathbb{F}G(x^2+1)$. Observe that G is a semi-direct product of the normal closure N of X and the infinite cyclic group Y generated by y. Furthermore, N is isomorphic to $\mathbb{Z}[1/2]$. This means that every element of G can be written as a product $y^k x^q$, with $k \in \mathbb{Z}$ and $q \in \mathbb{Z}[1/2]$ uniquely determined.

We now claim that the group ring $\mathbb{F}G$, viewed as a Y-module, decomposes as a direct sum of I and $J := \bigoplus_{0 \le q < 1} \mathbb{F}Yx^q$. To show that I and J intersect trivially, consider for $(k,q) \in \mathbb{Z} \times \mathbb{Z}[1/2]$ the functions $\lambda_{k,q} \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}G, \mathbb{F})$, defined by

$$\lambda_{k,q}(y^{k'}x^{q'}) = 1 \quad \Leftrightarrow \quad k = k' \text{ and } q - q' \in \mathbb{Z}.$$

If $r = s(y+x+1) + t(x^2+1)$ for some $s, t \in \mathbb{F}G$, then a simple calculation shows that

$$\lambda_{k,q}(ry^{-1}) = \lambda_{k,q}(s(y+x+1)y^{-1}) = \lambda_{k,q}(s) + \lambda_{k+1,2q}(s).$$

Here we assume that $r \in J$, then $\lambda_{k,q}(ry^{-1}) = 0$ for $1/2 \le q < 1$, and

$$\lambda_{k,q}(s) = \lambda_{k+1,2q}(s) \quad \text{for } 1/2 \le q < 1.$$

Therefore $\lambda_{k,q}(s)$ either vanishes for all pairs (k,q) or is non-zero for an infinite number of pairs (k,q). Since the latter is impossible, it follows that $\lambda_{k,q}(ry^{-1}) = 0$ for all k and q, and therefore $r = 0$.

Secondly, observe that for every $q \in \mathbb{Z}[1/2]$ there is a q' with $0 \le q' < 2$ such that $y^k x^q \equiv y^k x^{q'} \pmod{I}$. If $1 \le q < 2$, then

$$y^k x^q \equiv y^k x^{q-1}(y+1) = y^{k+1} x^{q-(2-q)} + y^k x^{q-1} \pmod{I}.$$

Repeating this process, if necessary, we can show that any element of $\mathbb{F}G/I$ can be represented by an element of J, and our claim follows.

We have now proved that D_G is free of infinite rank as a Y-module. In view of Lemma 4.1 this means that $\tilde{e}(G, Y) = 1$. On the other hand, the element $y^k x^q + I$ of D_G is stabilized by $x^{2^{1-k}}$, which implies that D_G is locally finite as an X-module and $\tilde{e}(G, X) = \infty$.

Let S be any subgroup of G. The equivalence (0) is proved in Lemma 2.4(ii), and if S is trivial, then (1) holds. Now suppose that S is a non-trivial subgroup of infinite index in G. If $S \cap X$ is non-trivial, then

$$\tilde{e}(G, S) \ge \tilde{e}(G, S \cap X) = \tilde{e}(G, X) = \infty.$$

Otherwise S intersects N trivially, so S must be infinite cyclic and generated by some element $y^k x^q$ with $k \ge 1$. Let Y_k denote the cyclic group generated by y^k and $G_k : Y_k N$. It is easy to find an automorphism of G_k which maps S to Y_k, therefore

$$\tilde{e}(G, S) = \tilde{e}(G_k, S) = \tilde{e}(G_k, Y_k) = \tilde{e}(G, Y) = 1.$$

This completes the proof of our proposition. □

We conclude this section with examples of pairs (G, S) for which e and $\tilde{e}$ take values other than 0, 1, 2 or infinity. The first example is due to Scott [12].

Proposition 4.6. *Let G be the fundamental group of a closed surface F and let S be the fundamental group of a compact incompressible subsurface X of F. Then $e(G,S)$ equals the number of boundary components of X.* □

Observe that by the above remark $\tilde{e}(G,S)$ equals 0, 2 or infinity for all these pairs.

As an example for the next proposition let G_1 be a knot group and S the free abelian subgroup of rank two generated by the meridian and a longitude of the knot. If the knot is prime, then $(G_1, \{S\})$ is a PD_3-*pair* in the sense of Bieri and Eckmann [2]. If the knot space is also hyperbolic, then it follows from [13, Proposition 4.5(ii)] that S is malnormal in G_1, which means that $S \cap S^g = 1$ for all $g \in G_1 \setminus S$.

Proposition 4.7. *Suppose that $G = G_1 *_S G_2$ is a free product with amalgamation, and*
(a) $(G_1, \{S\})$ *is a* PD^3-*pair*,
(b) S *is free abelian of rank two*,
(c) S *is malnormal in* G_1,
(d) G_2 *is free abelian of rank two and S has index n in* G_2.
Then $\tilde{e}(G,S)$ equals n.

Before we can prove this result we need some more information about the embedding of S in G.

Lemma 4.8. *Under the hypothesis of Proposition* 4.5 *the following hold*:
(i) $G_2G_1 = \{g \in G \mid S^g \cap G_1 > 1\}$.
(ii) $G_2 = \{g \in G \mid S^g \cap G_2 > 1\}$.

Proof. This can be seen most easily by considering the tree Y which corresponds to the splitting of G over S, according to the theory of Bass and Serre [4]. This is an oriented tree which admits a left action of G, such that G acts transitively on the edges of Y and fixed-point freely on the vertices. Furthermore, there is an edge e of Y such that e, its initial vertex ιe and its terminal vertex τe have stabilizers S, G_1 and G_2, respectively.

Now suppose that x is an element of $S^g \cap G_1$ for some element g of G. Then x stabilizes ιe and $g^{-1}e$, and therefore x also stabilizes the geodesic γ joining ιe to $g^{-1}e$. Condition (c) says that for any translate $h\iota e$ at most one of the edges adjacent to $h\iota e$ can be stabilized by x. Therefore γ can meet the orbit of ιe only in its end points. This means that γ contains at most two edges and $g^{-1} = g_1g_2$ for some elements g_i of G_i, thus (i) holds. A similar but simpler argument proves (ii). □

Proof of Proposition 4.7. Consider the Mayer–Vietoris sequence of $G = G_1 *_S G_2$ with coefficients in $\mathcal{F}_S G$.

$$(\mathcal{F}_S G)^G \to (\mathcal{F}_S G)^{G_1} \oplus (\mathcal{F}_S G)^{G_2} \to (\mathcal{F}_S G)^S \to H^1(G, \mathcal{F}_S G)$$
$$\to H^1(G_1, \mathcal{F}_S G) \oplus H^1(G_2, \mathcal{F}_S G) \to \cdots .$$

The proposition will be verified when we show that these groups are

$$0 \to 0 \oplus \mathbb{F} \to \mathbb{F}^n \to \mathbb{F}^{\tilde{e}(G,S)-1} \to 0 \oplus 0 \to \cdots.$$

For any subgroup T of G the module $\mathscr{F}_S G$, viewed as a T-module, decomposes as a direct sum

$$\mathscr{F}_S G \cong \bigoplus_{g \in S \backslash G/T} \mathscr{F}_S(SgT).$$

The invariant submodule $(\mathscr{F}_S(SgT))^T$ is non-trivial if and only if $|T : S^g \cap T| < \infty$.

Now $(\mathscr{F}_S G)^G = 0$ since S has infinite index in G. By Lemma 4.6(i) we have $(\mathscr{F}_S G)^{G_1} = (\mathscr{F}_S G_1)^{G_1} = 0$, and by the second part $(\mathscr{F}_S G)^{G_2} = (\mathscr{F}_S G_2)^{G_2} \cong \mathbb{F}$ and $(\mathscr{F}_S G)^S = (\mathscr{F}_S G_2)^S \cong \mathbb{F}^n$.

Using Poincaré duality for G_2, we obtain

$$H^1(G_2, \mathscr{F}_S G) \cong H_1(G_2, \mathscr{F}_S G) \cong \bigoplus_{g \in S \backslash G/G_2} H_1(S^g \cap G_2, \mathscr{P}S_g).$$

By Lemma 4.6(ii) the intersection $S^g \cap G_2$ is non-trivial only if $g \in G_2$, but then S commutes with g and we have

$$H_1(S^g \cap G_2, \mathscr{P}S_g) = H_1(S, g\mathscr{P}S) \cong H^1(S, g\mathscr{P}S) = 0.$$

Therefore $H^1(G_2, \mathscr{F}_S G) = 0$, and a similar argument shows that $H^1(S, \mathscr{F}_S G) = 0$.

Finally, we have to show that $H^1(G_1, \mathscr{F}_S G) = 0$. To prove this, consider the long exact sequence of the pair $(G_1, \{S\})$.

$$H^0(G_1, \mathscr{F}_S G) \to H^0(S, \mathscr{F}_S G) \to H^1(G_1, S; \mathscr{F}_S G) \to H^1(G_1, \mathscr{F}_S G)$$
$$\to H^1(S, \mathscr{F}_S G).$$

The first and last term of this sequence vanish, and $H^0(S, \mathscr{F}_S G) \cong \mathbb{F}^n$. Using the relative version of Poincaré duality, we find

$$H^1(G_1, S; \mathscr{F}_S G) \cong H_2(G_1, \mathscr{F}_S G) \cong \bigoplus_{g \in S \backslash G/G_1} H_2(S^g, \mathscr{P}Sg) \cong \mathbb{F}^n.$$

Therefore $H^1(G_1, \mathscr{F}_S G) = 0$, and the proof is completed. □

Note added in proof. The commensurizer has also been introduced by L. Corwin, Proc. Amer. Math. Soc. 47 (1975) 279–287, in the context of representation theory.

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